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Ergodic Properties of a Simple Deterministic Traffic Flow Model

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We study statistical properties of a family of maps acting in the space of integer valued sequences, which model dynamics of simple deterministic traffic flows. We obtain asymptotic (as time goes to infinity) properties of trajectories of those maps corresponding to arbitrary initial configurations in terms of statistics of densities of various patterns and describe weak attractors of these systems and the rate of convergence to them. Previously only the so called regular initial configurations (having a density with only finite fluctuations of partial sums around it) in the case of a slow particles model (with the maximal velocity 1) have been studied rigorously. Applying ideas borrowed from substitution dynamics we are able to reduce the analysis of the traffic flow models corresponding to the multi-lane traffic and to the flow with fast particles (with velocities greater than 1) to the simplest case of the flow with the one-lane traffic and slow particles, where the crucial technical step is the derivation of the exact life-time for a given cluster of particles. Applications to the optimal redirection of the multi-lane traffic flow and a model of a pedestrian going in a slowly moving crowd are discussed as well.

KEY WORDS: Dynamical system; traffic flow; substitution dynamics; attractor; rate of convergence; large deviations.

1. INTRODUCTION

Let $X_M := \{x = (\dots x_{-1}x_0x_1\dots) : x_i \in \mathscr{A}_M, i \in \mathbb{Z}\}$ be the space of bi-infinite sequences (which we also call configurations) from the alphabet $\mathscr{A}_M = \{0, 1, 2, \dots, M\}$. We equip this space with the metric

$$\mathrm{dist}_M(x, y) := \sum_{i=-\infty}^{\infty} (M+1)^{-|i|} |x_i - y_i|$$

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and consider a map $T_{1,M}: X_M \to X_M$ from this metric space into itself:

$$(T_{1,M}x)_i := x_i + \min\{x_{i-1}, M - x_i\} - \min\{x_i, M - x_{i+1}\}. \qquad (1.1)$$

One can interpret the ith coordinate of $x \in X_M$ as x_i particles and $M - x_i$ holes (empty places) located at the site i of the integer lattice $\mathbb{Z}$. Then this map can be considered as a discrete time/discrete space model for multi-lane highway traffic when a particle (vehicle) at site i of the lane j can switch to any other lane j' (nonnecessary neighboring) whenever it does not disrupt the motion of other particles, i.e., the sites $i, i+1$ of the lane j' are not occupied. From the point of view of probability theory the dynamics of this map is a deterministic version of an asymmetric exclusion process, i.e., the motion of a collection of random walkers constrained to the nonintersection assumption (see, e.g., refs. 7 and 11). Traffic flow phenomena have attracted considerable interest during last three decades both from the applied and theoretical points of view. For the general account on these matters we refer the reader to recent reviews[5, 9] (and numerous references cited there). The deterministic traffic flow models considered in this paper are not new (except probably for the case $v, M > 1$) and have been studied extensively both on numerical and theoretical levels (see, e.g., refs. 3, 5 and 13–15) in the case of spatially periodic initial configurations, and from the point of view of the evolution of random initial data using various statistical physics approaches (see, e.g., refs. 1, 2, and 10). We shall show that a much more detailed description of asymptotic (as time goes to infinity) properties of the system on the level of individual trajectories is available and shall concentrate only on the mathematical background of deterministic models of traffic flows.

We shall refer to the system $(T_{1,M}, X_M)$ as the *slow particles model*, and to take into account traffic flows where particles can move with the (maximal) velocity $v > 1$ (a *fast particles model*) we consider a family of maps $T_{v,M}: X_M \to X_M$ describing the M-lane traffic flow model with the maximal velocity $|v|$, i.e., a particle in this flow can move to the right (left if $v < 0$) by at most $|v|$ positions if those positions are not occupied. Note that in the case when both parameters v and M are greater than 1 an additional technical step consisting in a "sawtooth redirection" S of particles from the configuration x between M lanes (see Section 4) is needed to give a concise definition of the multi lane fast particles model $T_{v,M}x := \sum_j T_{v,1}(S_0 x)^{(j)}$.

To simplify the notation we shall drop the indices if they are equal to 1, i.e., T_2 means the case $v = 2$, $M = 1$ and $T_{1,3}$ means the case $v = 1$, $M = 3$, while T stands for the case $v = M = 1$. By a *dual* configuration for the configuration $x \in X_M$ we mean a configuration $x^* \in X_M$ such that $x_i^* = M - x_i$ $\forall i$. The operation of taking a dual can be applied also to the map by means of the relation: $T_{v,M}^* x^* := (T_{v,M}x)^*$ $\forall x \in X_M$.

To illustrate the usage of the dual operation consider the slow particles model with "smart drivers," who anticipating the motion of at most m cars ahead, may move to an occupied site ahead of it with the maximal velocity 1. Example for the case $M=2$: $\langle 01110\rangle \to \langle 01011\rangle$, where $\langle \cdot \rangle$ denotes the main period of a (space) periodic configuration. It is straightforward to show that this model is described by the map $\tilde{T}x = (T_{-m,1}x^*)^* = T^*_{-m,1}x$.

By a *word* A we shall call any (finite or infinite) sequence of elements $a_i \in \mathscr{A}_M$ and introduce the notion of the *density* of a finite word B in a finite word A as

$$\rho(A, B) := \frac{1}{|A|} \sum_{i=1}^{|A|-|B|+1} \min_{j=1}^{|B|} \left\{ \left\lfloor \frac{A_{i-1+j}}{B_j} \right\rfloor \right\}, \tag{1.2}$$

where $|A|$ is the length of the word A, $A_j \in \mathscr{A}_M$ is the jth element of the word A, $\lfloor a \rfloor$ is the integer part of the number a if $|a| < \infty$ and is equal to 0 otherwise, and we set $0/0 \equiv 1$ here and in the sequel. In the case $M=1$ the number $\rho(A, B) \in [0, 1]$ and is equal to the number of occurrences of the subword B in A divided by the length of A, while in the general case $M > 1$ we have $\rho(A, B) \in [0, M]$ and the formula (1.2) takes into account multiplicities of those occurrences. Example: $\rho(255, 12) = \frac{1}{3}(2+2) = 4/3$.

The generalization of this notion for an infinite word/configuration $x \in X_M$ leads to the notion of *lower/upper density*:

$$\rho_\pm(x, A) := \lim_{n,m\to\infty} \binom{\sup}{\inf} \rho(x[-n, m], A),$$

where (and in the sequel) lim sup corresponds to the index $+$ and lim inf to the index $-$, and $x[n, m]$ is a subword of the word x which starts from the position n and goes till the position m in the original word. The asymmetry with respect to n and m is necessary to take into account the possibility to have left and right "tails" with different statistics: for $x := \ldots 00001111\ldots$ we have $\rho_-(x, 1) = 0$ and $\rho_+(x, 1) = 1$, while $\rho(x[-n, n], B) \xrightarrow{n\to\infty} 1/2$. Observe also that for a (space) periodic configuration $\langle A\rangle := \ldots AAA\ldots$ we have $\rho_-(\langle A\rangle, B) = \rho_+(\langle A\rangle, B) \equiv \rho(\langle A\rangle, B)$ for any pair of finite words A, B.

For a collection of particles on a lattice one can define its average velocity as follows. For each particle in a configuration $x \in X$ we define its "local" velocity as a distance by which the particle will move on the next step of the dynamics, and, since for $M > 1$ a site i in the configuration $x \in X_M$ may contain several particles (i.e., $x_i > 1$), we sum up their velocities to get the quantity $V(x, i)$. If $x_i = 0$ we set $V(x, i) := 0$. For example, in the case of the map $T_{1,M}$ we have $V(x, i) := \min\{x_i, M - x_{i+1}\}$. Now we define the lower/upper *average velocity* as

$$V_{\pm}(x) := \lim_{n,m\to\infty} \binom{\sup}{\inf} \frac{1}{\rho(x[-n,m],1)\cdot(n+m+1)} \sum_{i=-n}^{m} V(x,i).$$

Often it is more convenient to work with another statistics, called *flux*, equal to the number of particles crossing a given position on the lattice per unit time, i.e., $\Phi(x[-n,m]) := \frac{1}{n+m+1}\sum_{i=-n}^{m} V(x,i)$. Thus we define the upper/lower *average flux* as

$$\Phi_{\pm}(x) := \lim_{n,m\to\infty} \binom{\sup}{\inf} \frac{1}{n+m+1} \sum_{i=-n}^{m} V(x,i).$$

We shall use also the notation $\Phi_{\pm}^{(v)}$ to indicate the maximum velocity if needed, and $0_i := \underbrace{00\cdots0}_{i}$. The connection of the flux to the densities is given by the following simple result.

Lemma 1.1. $\Phi_{\pm}^{(v)}(x) = \sum_{i=1}^{v} \rho_{\pm}(x, 10_i)$ for $x \in X$, in particular $\Phi_{\pm}^{(1)}(x) := \rho_{\pm}(x, 10)$.

Proof. By definition we have

$$\begin{aligned}
\Phi_{\pm}^{(v)}(x) &= v\cdot\rho_{\pm}(x,10_v) + (v-1)\cdot(\rho_{\pm}(x,10_{v-1}) - \rho_{\pm}(x,10_v)) \\
&\quad + \cdots + 1\cdot\rho_{\pm}(x,10) \\
&= (v-(v-1))\cdot\rho_{\pm}(x,10_v) + ((v-1)-(v-2))\,\rho_{\pm}(x,10_{v-1}) \\
&\quad + \cdots + 1\cdot\rho_{\pm}(x,10) \\
&= \sum_{i=1}^{v} \rho_{\pm}(x,10_i). \qquad \blacksquare
\end{aligned}$$

The main results of the paper are the following statements.

Theorem 1.2 (Invariance of Densities). $\rho_{\pm}(T_{v,M}^{t}x, A) = \rho_{\pm}(x, A)$ for all $x \in X_M$ and $t \in \mathbb{Z}_+$ if and only if $A \in \{0, 1\}$.

Denote by $\mathrm{Free}_v := \{x \in X_M : V(x,i) = v\cdot x_i\ \forall i \in \mathbb{Z}\}$ the subset of configurations where all particles have the maximal available velocity and thus move independently. Clearly, $T_{v,M}(\mathrm{Free}_v) = \mathrm{Free}_v$ and $T_{v,M}(\mathrm{Free}_v^*) = \mathrm{Free}_v^*$.

Theorem 1.3 (Convergence). The set $\mathrm{Free}_v \cup \mathrm{Free}_v^*$ is the only locally maximal weak attractor of the dynamical system $(T_{v,M}, X_M)$, and for $x \in X_M$ we have

$$T_{v,M}^t x \xrightarrow{t \to \infty} \begin{cases} \mathrm{Free}_v & \text{if} \quad \rho_+(x,1) \leqslant \frac{M}{v+1} \\ \mathrm{Free}_v^* & \text{if} \quad \rho_-(x,1) \geqslant \frac{M}{v+1}. \end{cases}$$

Theorem 1.4 (Limit Flux). $\Phi_\pm^{(v)}(T_{v,M}^t x) \xrightarrow{t \to \infty} F_{v,M}(\rho_\pm(x,1))$, where

$$F_{v,M}(\xi) := \begin{cases} v\xi & \text{if} \quad \xi \leqslant \frac{M}{v+1} \\ M - \xi & \text{otherwise.} \end{cases}$$

The function $F_{v,M}$, describing the limit average flux and the corresponding function for the limit average velocity (often called a fundamental diagram in the literature) are shown in Fig. 1.

Denote by μ_p a product (Bernoulli) measure with the density pM on the space of sequences X_M.

Theorem 1.5 (Typical Dynamics). For μ_p-a.a. $x \in X_M$ we have $\rho(x,1) = pM$ and $\mathrm{dist}_M(T_{v,M}^t x, \mathrm{Free} \cup \mathrm{Free}^*) \leqslant M^{-t/\gamma+1}$ and $\limsup_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n} V(T_{v,M}^n x, 1) = F_{v,M}(\rho(x,1))$ for any $\gamma \in (0,1)$.

In the last statement one can use instead of μ_p any probabilistic translation invariant measure with fast enough decay of correlations (see Lemma 2.11).

Proofs of Theorems 1.2–1.5 are based on the reduction of the general case $v, M \geqslant 1$ to the simplest one $v = M = 1$. For $v = 1, M > 1$ this reduction boils down to the proof that a multi lane traffic flow can be represented by a direct product of one-lane flows (see Theorem 4.1 describing the "sawtooth redirection" construction). In the case $v > 1$, $M = 1$ we make use of a specially constructed substitution dynamics (see Lemma 3.1) to

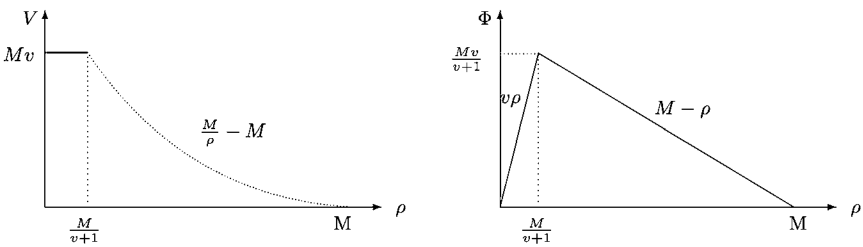

Fig. 1. Fundamental diagram for $T_{v,M}$: dependence of the average velocity V or the flux Φ on the density $\rho = \rho(\cdot, 1)$.

prove the reduction, while in the general case $v, M > 1$ we combine these two arguments. The main technical step of the analysis in the case $v = M = 1$ is the derivation of the exact life-time for a given cluster of particles, i.e., the number of iterations after which it will disappear, described in Lemma 2.4. Note that earlier only very weak (and unnaturally large) estimates of the life-time type were known (see, e.g., refs. 3, 4, and 8). We provide also the analysis of the rate of convergence to the limit of various statistics for "typical" initial configurations based on large deviations estimates (see end of Section 2), and study the dynamics of a passive tracer in the flow of fast particles imitating a motion of a fast pedestrian in a slowly moving crowd of people (Section 6).

It is clear that the main problem in the study of traffic flows is the analysis of "traffic jams" (without them the dynamics is trivial): we shall say that a segment $x[n, m]$ with $m > n$ corresponds to the *jammed cluster* if for each particle belonging to this segment either its velocity is strictly less than the maximum available velocity v, or there is another particle at the same site for which this inequality holds. Note that in the case $v = M = 1$ the jammed cluster is the same as the cluster of particles.

2. THE ONE LANE SLOW PARTICLES MODEL (T, X)

This model has been introduced originally in ref. 14 for the case of a traffic flow on a finite lattice (say of size L) with periodic boundary conditions and studied numerically in a large number of publications. It is straightforward to show that this case corresponds to the restriction of the map T to (space) L-periodic configurations. The first "quasi"-analytic result for the L-periodic case has been obtained in ref. 8 for "typical" initial configurations of length L. However the first complete proof appeared only in ref. 3, where regular initial configurations on the infinite lattice (having a density with only finite fluctuations of partial sums around it) were considered as well. In this section we shall study the problem for all initial configurations, using a rather different and more simple approach than the one in ref. 3.

Let us start from the analysis of lower and upper densities. Note that if the lower density coincides with the upper one, i.e., the limit value exists, we call this common value the density $\rho(\cdot, \cdot)$. Example when they do not coincide: $1 = \rho_+(\ldots 111000\ldots, 1) \neq \rho_-(\ldots 111000\ldots, 1) = 0$.

Lemma 2.1. $\rho_\pm(x, 1) = 1 - \rho_\mp(x, 0)$, and thus $\rho_\pm(x^*, 1) = 1 - \rho_\mp(x, 1)$.

Proof. By the definition of the lower density we have

$$\begin{aligned}\rho_-(x,1) &= \liminf_{n,m\to\infty} \rho(x[-n,m],1)\\ &= 1-\limsup_{n,m\to\infty} \rho(x[-n,m],0)\\ &= 1-\rho_+(x,0),\end{aligned}$$

since $\rho(A,1)\cdot|A|+\rho(A,0)\cdot|A|=|A|$ for any finite binary word A. The derivation for the upper density follows the same argument, while the second statement follows from the identity: $\rho(x^*[-n,m],1)=\rho(x[-n,m],0)=1-\rho(x[-n,m],1)$. ▮

Lemma 2.2. $\rho_\pm(x,A)\geqslant\rho_\pm(x,B)\cdot\rho(B,A)$ for any configuration $x\in X$ and any pair of finite words A, B.

Proof. If $A\not\subseteq B$ the inequality becomes trivial, since $\rho(B,A)=0$, while $\rho_\pm(x,A)\geqslant 0$. Assume now that $A\subseteq B$. Then $\forall n,m\in\mathbb{Z}$ we have $\rho(x[-n,m],A)\geqslant\rho(x[-n,m],B)\cdot\rho(B,A)$ because the right hand side takes into account only those enclosures of B to x when the word B belongs to a segment $x[i,j]=A$, while there might be other enclosures as well. ▮

Proof of the Necessity Part of Theorem 1.2 in the Case v=M=1. Let us prove first that $\rho_\pm(x,1)=\rho_\pm(Tx,1)$ for all $x\in X$.

For any $n,m\in\mathbb{Z}_+$ we have $|\sum_{i=-n}^{m}(x_i-(Tx)_i)|\leqslant 2$, since during one iteration of the map at most one particle can enter the interval of sites from $-n$ to m (from behind) and at most one particle can leave this interval.

By the definition of the lower density there is a sequence of pairs $(n_j,m_j)\xrightarrow{j\to\infty}(\infty,\infty)$ such that

$$\frac{1}{n_j+m_j+1}\sum_{i=-n_j}^{m_j} x_i \xrightarrow{j\to\infty} \rho_-(x,1).$$

On the other hand, since $|\sum_{i=-n_j}^{m_j} x_i-\sum_{i=-n_j}^{m_j}(Tx)_i|\leqslant 2$, we deduce that $\rho_-(x,1)$ is a limit point for partial sums for the sequence Tx. Therefore we need to show only that this is indeed the lower limit. Assume, on the contrary, that there is another limit point, call it ξ, for the partial sums for Tx such that $\xi<\rho_-(x,1)$. Doing the same operations with the partial sums for Tx converging to ξ we can show that this value is also a limit point for the partial sums for the sequence x, and, hence, ξ cannot be smaller than $\rho_-(x,1)$.

The proof for the upper density follows from the same argument.

By Lemma 2.1 we have $\rho_\pm(x, 1) = 1 - \rho_\mp(x, 0)$, which proves the preservation of the density of zeros as well.

To prove that all other statistics are not preserved under dynamics we need to study it in more detail. Therefore we postpone the continuation of the proof till the end of this section.

Lemma 2.3. $T^* = T_{-1}$.

Proof. The action of the map T on binary configuration is equivalent to the exchange of any pair 10 to 01. Since the dual map describes the dynamics of holes it corresponds in this case to the exchange of pairs 01 to 10, which proves the statement. ∎

By a *cluster* (of particles) in a binary configuration $x \in X$ we mean a subword $x[n, m]$ with $n < m$ such that $x_i = 1 \ \forall i \in \{n,..., m\}$ and $x_{n-1} = x_{m+1} = 0$. After each iteration of the map T the last particle in the cluster moves away (i.e., $(Tx)_m = 0$) and either appears a new element in the cluster from the left (i.e., $(Tx)_{n-1} = 1$ and $(Tx)_{n-2} = 0$), or the first particle preserves its position n. Therefore the number of particles in a given cluster cannot increase, and the time up to the moment when the cluster length shrinks to 1 (i.e., it disappears) we shall call the *life-time* of the cluster or the number of iterations which are needed for a given cluster to disappear.[2]

Define an integer-valued function

$$I(x, i) := \max\{k < i : \rho(x[k, i], 1) = \rho(x[k, i], 0)\} \tag{2.1}$$

and consider a collection of sets $\Omega^{2n} := \{A \in \{0, 1\}^{2n} : A_{2n} = 1, \ I(A, 2n) = 1\}$. Note that by the relation (2.1) for any $A \in \Omega^{2n}$ we have $\rho(A, 1) = |A|/2$. Observe also that if $A \in \Omega^{2n}$ then for any $0 < m < n$ and any word $B \in \{0, 1\}^{2m}$ such that $B_i = A_i \ \forall 0 < i \leqslant 2m$ we have $B \notin \Omega^{2m}$. Therefore we shall call the words from Ω^{2n} *minimal* words (or minimal intervals) corresponding to clusters of particles in their ends.

Lemma 2.4.[3] Let $\sum_{i=k}^{m} x_i = m - k + 1$, $x_{k-1} = x_{m+1} = 0$ (i.e., the positions from k to m correspond to a cluster of particles) and let $I(x, m) > -\infty$. Then after exactly $\frac{1}{2}(m - I(x, m) - 1)$ iterations (which is

[2] Observe that speaking about the dynamics of a given cluster of particles we take care only about the particles remaining in the cluster (including those that joined it) but not about the particles which already left it and, in principle, might join another cluster.

[3] Technical estimates of this type were considered also in refs. 1, 4, and 10.

equal to the number of ones minus one in the word $x[m-I(x,m),m]$) this cluster will disappear. If $\rho_+(x,1) \leqslant 1/2$ then $\forall i \in \mathbb{Z}$ we have $I(x,i) > -\infty$.

Proof. Let $n = I(x,m)$. Consider a map $\Gamma\colon \Omega^{2n} \to \mathbb{Z}^{2n-2}$ defined by the relation:

$$(\Gamma A)_{i+1} = A_i + \min\{A_{i-1}, 1-A_i\} - \min\{A_i, 1-A_{i+1}\}.$$

Observe that this is a shift to the right of the action of our map T. We shall prove that for each n we have $\Gamma\colon \Omega^{2n} \to \Omega^{2n-2}$ (see Fig. 2).

Let $A \in \Omega^{2n}$ and let ζ_A be the position of the last 0 in A. For each word $A \in \Omega^{2n}$ define a new word $A' \in \mathbb{Z}^{2n-2}$ as follows:

$$A'_i := \begin{cases} A_i & \text{if} \quad i \leqslant 2n-2 \quad \text{and} \quad i \neq \zeta_A \\ 1 & \text{if} \quad i = \zeta_A. \end{cases}$$

Then $|A'| = |A| - 2 = 2n-2$, $\rho(A',1) = 1/2$ and thus $A' \in \Omega^{2n-2}$ since otherwise A would be not minimal as well. Now using the following simple identity:

$$(\Gamma A)_i = \begin{cases} (\Gamma A')_i & \text{if} \quad \zeta_A - i \notin \{1,2\} \\ 0 & \text{if} \quad i = \zeta_A - 2 \\ A_{\zeta_A} & \text{if} \quad i = \zeta_A - 1 \\ 1 & \text{if} \quad i = |A| - 2, \end{cases}$$

we get $\Gamma A \in \Omega^{2n-2}$. Example: $A = 001011$, $A' = 0011$.

It remains to show that if we have a cluster of particles located in the end of a minimal configuration $A \in \Omega^{2n}$ then this cluster (i.e., particles at sites from $\zeta_A + 1$ to $2n$) will vanish after $n-1$ iterations. Observe that after one iteration of the map T the cluster either preserves its length, or the

$$\begin{array}{lcccccccccc}
A = & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\Gamma^1 A = & & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & \\
\Gamma^2 A = & & & 0 & 0 & 1 & 0 & 1 & 1 & & \\
\Gamma^3 A = & & & & 0 & 0 & 1 & 1 & & & \\
\Gamma^4 A = & & & & & 0 & 1 & & & &
\end{array}
\qquad
\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
& 0 & 0 & 1 & 0 & 1 & 1 & \\
& & 0 & 0 & 1 & 1 & & \\
& & & 0 & 1 & & &
\end{array}$$

Fig. 2. Examples of the action of Γ^t on Ω^{10} (left) and Ω^8 (right).

length decreases by one (when two positions immediately preceding the cluster are occupied by two zeros). The map Γ defined above controls this process since for each $A \in \Omega^{2n}$ and each $0 \leqslant t \leqslant n-1$ the last positions starting from $(\zeta_{\Gamma^t A}+1)$ correspond to the cluster under study. ▮

Corollary 2.5. The dynamics of a cluster of particles depends only on the distribution of particles lying below the cluster. Moreover, for a given cluster of particles only particles belonging to its minimal word can join the cluster. Thus, if a particle does not belong to any minimal word, then for each $t \in \mathbb{Z}_+ \cup \{0\}$ its local velocity in the configuration $T^t x$ is equal to 1, i.e., it moves freely.

Lemma 2.6. Let $A := x[n, m]$ and $A' := x[n', m']$ be two minimal words in the configuration $x \in X$. Then the inequality $m < m'$ yields either $m < n'$ (i.e., $A \cap A' = \emptyset$), or $n' < n$ (i.e., $A \subset A'$).

Proof. Assume on the contrary that $n \leqslant n' \leqslant m$. Then by the definition of a minimal word we have

$$\begin{aligned}\tfrac{1}{2}(m'-n'+1) &= \rho(x[n', m'], 1)\cdot(m'-n'+1)\\ &= \rho(x[n', m], 1)\cdot(m-n'+1)+\rho(x[m, m'], 1)\cdot(m'-m+1)\\ &> \tfrac{1}{2}(m-n'+1+m'-m+1)=\tfrac{1}{2}(m'-n'+2).\end{aligned}$$

We came to a contradiction. ▮

Corollary 2.7. For a given particle ξ in a configuration $x \in X$ let the length of the largest minimal word to which ξ belongs be $2n$. Then for any $t \geqslant n-1$ the local velocity of the particle ξ in the configuration $T^t x$ is equal to 1.

Observe that in the case $v = M = 1$ the set $\text{Free} = \{x \in X_1 : x_i x_{i+1} = 0 \ \forall i\}$ is the union of "free" particles (i.e., particles having velocity 1), while its dual $\text{Free}^* = \{x \in X_1 : (1-x_i)(1-x_{i+1}) = 0 \ \forall i\}$ corresponds to "free" holes (i.e., to holes having velocity -1).

Lemma 2.8. Let $\rho_+(x, 1) \leqslant 1/2$, then each cluster of particles in the configuration x will disappear after a finite number of iterations and $\text{dist}(T^t x, \text{Free}) \xrightarrow{t \to \infty} 0$. If there is a cluster of particles having an infinite minimal word (i.e., which does not vanish in finite time), then $\rho_+(x, 1) > 1/2$. If $\rho_+(x, 1) > 1/2$ then there are clusters of particles with arbitrary large (but may be finite) minimal words.

Proof. Let $x \in X$ satisfy the assumption that $\rho_+(x, 1) \leqslant 1/2$ and let the segment $x[n, m]$ be a cluster of particles. Then there exist a pair of integers n', m' such that $-\infty < n' < n < m \leqslant m' < \infty$ and $x[n', m']$ is the largest minimal word covering the cluster of particles $x[n, m]$ (otherwise this would contradict to the definition of the upper density). Hence by Lemma 2.4 after at most $(m'-n')/2$ iterations this cluster will disappear and all particles will become free. Since this argument can be applied to any cluster of particles, this yields the first statement.

Assume now that the minimal word of a cluster of particles $x[n, m]$ is not bounded. Then for any $n' < n$ we have $x[n', m] > 1/2$ and thus for any $k \in \mathbb{Z}_+$ we have

$$\begin{aligned}\rho(x[n-k^2, m+k], 1) &\geqslant \frac{m+k-n+k^2+1}{m-n+k^2+1}\,\rho(x[n-k^2, m], 1)\\ &> \frac{m-n+k^2+1}{m+k-n+k^2+1}\,\frac{1}{2}\\ &= \left(1 - \frac{k}{m+k-n+k^2+1}\right)\frac{1}{2} \xrightarrow{k\to\infty} \frac{1}{2}.\end{aligned}$$

Therefore $\rho_+(x, 1) > 1/2$, which proves the second statement.

The last statement is an immediate consequence of the definition of the minimal word. ∎

Lemma 2.9. Let $\rho_-(x, 1) > 1/2$ then $\mathrm{dist}(T^{*t}x^*, \mathrm{Free}) \xrightarrow{t\to\infty} 0$.

Proof. By Lemma 2.1 we have $\rho_+(x^*, 1) = 1 - \rho_-(x, 1) < 1/2$. On the other hand, Lemma 2.3 shows that asymptotic properties of the maps T and T^* coincide, thus we can apply the statements of Lemma 2.8 for the case of $(T^*)^t x^*$ to prove the desired result. ∎

Note now that there are configurations not satisfying the assumptions of Lemmas 2.8 and 2.9 which still converge to $\mathrm{Free} \cup \mathrm{Free}^*$ under the action of the map T. Indeed, let $y = \ldots 111000\ldots$ and let the index 0 correspond to the first 0 in y. Observe that $\rho_-(y, 1) = 0 < \rho_+(y, 1) = 1$, however $\mathrm{dist}(T^t y, \mathrm{Free}) = 2^{-(t+1)}(1 + 2^{-2} + 2^{-4} + \cdots) = \frac{2}{3}\cdot 2^{-t} \xrightarrow{t\to\infty} 0$, since for large t the "central" part of $T^t y$ will be occupied only by free particles. On the other hand, for $y^* = \ldots 000111\ldots$ we have $\mathrm{dist}(T^t y^*, \mathrm{Free}) = 2^{-1} + 2^{-3} + \cdots + 2^{-2n+1} + \cdots = 1/3$ for each $t \in \mathbb{Z}^+$, while $\mathrm{dist}(T^{*t} y^*, \mathrm{Free}) = \frac{2}{3}\cdot 2^{-t} \xrightarrow{t\to\infty} 0$.

Lemma 2.10. For $x \in X$ we have $\mathrm{dist}(T^t x, \mathrm{Free}) \xrightarrow{t\to\infty} 0$ if and only if $\rho_+(x[1, \infty], 1) := \limsup_{n\to\infty} \rho(x[1, n], 1) \leqslant 1/2$. If additionally there exists a pair $n, m \in \mathbb{Z}_+$ such that $\rho(x[i, i+m-1], 1) \leqslant 1/2$ for each $i \geqslant n$ we have $\mathrm{dist}(T^t x, \mathrm{Free}) \leqslant \mathrm{Const}\, 2^{-t}$.

Proof. Observe that $\rho_+(x[1, \infty], 1) \leqslant 1/2$ implies that there exists $N \in \mathbb{Z}$ such that the life-time for each cluster of particles lying to the right from N is finite. On the other hand, the distance to the position N from the most right cluster of particles in $T^t x$ located to the left of N grows with t linearly. This proves the first statement and shows that the rate of convergence might be smaller than 2^{-t} only if the life-time of clusters of particles lying to the right from a sufficiently large position N is not bounded. The additional assumption guarantees that this cannot happen, which yields the second statement. ∎

Let $\mathcal{M}(X)$ be the set of probabilistic translation invariant measures on X and let $\mu[\phi(x)] := \int \phi(x)\, d\mu(x)$ for $\mu \in \mathcal{M}(X)$, in particular, $\mu[x_0] := \mu(x \in X : x_0 = 1)$. Consider a subset of $\mathcal{M}(X)$ corresponding to measures in the space of sequences with weak dependence between coordinates (exponentially fast decay of correlations):

$$\mathcal{M}_p(X) := \{\mu \in \mathcal{M}(X) : \mu[x_0] = p,\ |\mu[x_0 \cdot x_k] - \mu^2[x_0]| \leqslant C e^{-\alpha |k|}\}.$$

for some $C, \alpha > 0$ and $\forall k \in \mathbb{Z}$. Note that, e.g., a product (Bernoulli) measure $\mu_p \in \mathcal{M}_p(X)$.

Lemma 2.11. For any $\mu \in \mathcal{M}_p(X)$ we have $\rho(x, 1) = p$ for μ-a.a. $x \in X$, and thus $\mu(x \in X : \rho_-(x, 1) < 1/2 < \rho_+(x, 1)) = 0$.

Proof. Let $S_{n,m}(x) := \sum_{i=-n}^{m} x_i$. Then $\mu[S_{n,m}(x)] = p$ and by Chebyshev inequality $\forall \varepsilon > 0$ we have

$$\mu(x \in X : |S_{n,m}(x) - p| \geqslant \varepsilon) \leqslant \frac{1}{\varepsilon^2} \cdot \mu[(S_{n,m}(x) - p)^2].$$

A straightforward calculation shows that $\mu[(S_{n,m}(x) - p)^2] \leqslant \frac{C_1}{n+m+1}$ and thus

$$\mu\left(x \in X : \left|\frac{1}{n+m+1} \sum_{i=-n}^{m} x_i - p\right| \geqslant \varepsilon\right) \leqslant \frac{C_1}{(n+m+1)\, \varepsilon^2} \xrightarrow{n, m \to \infty} 0.$$

Therefore $\rho_\pm(x[-n, m], 1) \xrightarrow{n, m \to \infty} p$ in probability, which yields the existence of the density $\rho(x, 1) = p$ for μ-a.a. $x \in X$ and thus the statement under question. ∎

Denote by $\mathrm{Per}_n(T) := \{x \in X : T^n x = x\}$ the set of n-periodic (in time) trajectories of the map T and by $\mathscr{B}(Y) := \bigcup_{n \geqslant 0} T^{-n} Y$ the *basin of attraction* of a subset $Y \subset X$.

Lemma 2.12. $T \colon X \to X$ is a Lipschitz continuous map in the topology induced by the metrics $\mathrm{dist}(\cdot, \cdot)$. For each $n \in \mathbb{Z}_+$ there exists an n-periodic trajectory, and all periodic trajectories are unstable. $\mathrm{Clos}(\mathscr{B}(\mathrm{Free} \cup \mathrm{Free}^*)) = X$, $(\mathrm{Free} \cup \mathrm{Free}^*) \cap \mathrm{Per}_1(T) = \varnothing$, and $\mu_p(\mathscr{B}(\mathrm{Per}_1(T))) = 0$ while $\mathrm{Clos}(\mathscr{B}(\mathrm{Per}_1(T))) = X$.

Proof. Let us start with the Lipschitz continuity. Consider two configurations $x \neq y \in X$ and assume that $-n < 0$ is the largest negative index and $m \geqslant 0$ is the smallest nonnegative index of sites, where they differ, i.e., for all $-n < i < m$ we have $x_i = y_i$. Then we have

$$2^{-n} + 2^{-m} \leqslant \mathrm{dist}(x, y) \leqslant 2(2^{-n} + 2^{-m}).$$

On the other hand, a straightforward calculation shows that the closest to the origin from the left side differing coordinates of the configurations Tx and Ty belong to the set $\{-(n+1), -n, -(n-1)\}$, while the closest from the right side belong to $\{m-1, m, m+1\}$. Thus

$$2^{-(n+1)} + 2^{-(m+1)} \leqslant \mathrm{dist}(Tx, Ty) \leqslant 2(2^{-(n-1)} + 2^{-(m-1)}).$$

Therefore

$$\frac{1}{4} = \frac{2^{-(n+1)} + 2^{-(m+1)}}{2(2^{-n} + 2^{-m})} \leqslant \frac{\mathrm{dist}(Tx, Ty)}{\mathrm{dist}(x, y)} \leqslant \frac{2(2^{-(n-1)} + 2^{-(m-1)})}{2^{-n} + 2^{-m}} = 4.$$

For a given $n \in \mathbb{Z}_+$ consider a space-periodic configuration $x \in X$ with the (space) period n, e.g., $x_i = x_{i+n}$ $\forall i$. Then it is immediate to show that for any $t \in \mathbb{Z}$ the configuration Tx is again space periodic with the same period n and converges either to Free, or to Free*, depending on its density. This gives a construction of the n-periodic (in time) trajectories.

The structure on the set of fixed points $\mathrm{Per}_1(T)$ is a bit more involved:

$$\mathrm{Per}_1(T) := \left\{ x^{(n)} \in X : x_i^{(n)} = \begin{cases} 0 & \text{if } \; i < n \\ 1 & \text{otherwise} \end{cases} \right\}.$$

Indeed, assume that $Tx = x$, then either x does not have zero coordinates, or all coordinates starting from, say, nth, should be equal to one. Now for $x^{(n)} \in \mathrm{Per}_1(T)$ we define $y^{(n,m)} \in X$ such that

$$y_i^{(n,m)} = \begin{cases} 0 & \text{if } \; i < n, \text{ or } \; i > m \\ 1 & \text{otherwise} \end{cases}$$

for some $m > n$. Then $\mathrm{dist}(x^{(n)}, y^{(n,m)}) = 2^{-m} \xrightarrow{m\to\infty} 0$, while $\mathrm{dist}(T^t x^{(n)}, T^t y^{(n,m)}) \xrightarrow{t\to\infty} 2^{-(n-1)} \neq 0$. Thus for each $\varepsilon > 0$ there is a configuration $x' = x'(\varepsilon)$ such that $\mathrm{dist}(x^{(n)}, x') \leqslant \varepsilon$ and $T^t x' \not\to x^{(n)}$ as $t \to \infty$, which yields instability.

Observe now that the set $Y := X \setminus (\mathscr{B}(\mathrm{Free} \cup \mathrm{Free}^*)) = \{x \in X : \rho_-(x, 1) < 1/2 < \rho_+(x, 1)\}$ has μ_p-measure zero, since for each μ_p-typical trajectory the lower and upper densities coincide. Consider now an arbitrary configuration $x \in X$ and a sequence of configurations $\{y^{(n)}\}_n$ defined as

$$y_i^{(n)} = \begin{cases} x_i & \text{if } \ i < n, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\mathrm{dist}(x, y^{(n)}) \leqslant 2^{-n+1}$, on the other hand, $y^{(n)} \xrightarrow{n\to\infty} \mathrm{Per}_1(T)$, which proves the last statement. ▮

Remark. In the case of a finite cluster of particles, its last particle immediately leaves the cluster under dynamics. This is not the case for clusters not bounded from the right, which explains the existence of fixed points.

For a given reference measure μ_{ref} we shall say that a closed T-invariant set Y is a *weak attractor* if $\mu_{\mathrm{ref}}(\mathscr{B}(Y)) > 0$. A weak attractor Y is called a *Milnor* attractor if $\mu_{\mathrm{ref}}(\mathscr{B}(Y) \setminus \mathscr{B}(Y')) > 0$ for any proper compact invariant subset $Y' \subseteq Y$ (see, e.g., ref. 12).

Lemma 2.13. The set $\mathrm{Free} \cup \mathrm{Free}^*$ is a week attractor with respect $\mu_{\mathrm{ref}} = \mu_p$, but not a Milnor one, moreover it is not a topological attractor.

Proof. The sets Free and Free* are closed, since they contain all their limit points. Denote by $Z^{(p)} := \{x \in \mathrm{Free} : \rho(x, 1) = p\}$ the subset of the set Free containing only configurations with density p. The set $Z^{(p)}$ is T-invariant and by Lemma 2.11 we have $\mu_p(\mathscr{B}(Z^{(p)})) = 1$. Choose now an arbitrary single configuration from $Z^{(p)}$ and denote by $Z'^{(p)}$ the set consisting of this configuration and all its left and right space shifts. Clearly we have $\mu_p(Z^{(p)} \setminus (Z^{(p)} \setminus Z'^{(p)})) = \mu_p(Z'^{(p)}) = 0$. Observe that the points from the complement to the basins of attraction of Free and Free* are everywhere dense, which proves the absence of included open sets. The last statement follows from the fact that the basin of attraction does not contain any open set. ▮

Proof of Theorem 1.2 (Continuation). Let us prove now that for any word A with $|A| > 1$ the density $\rho_\pm(x, A)$ is not preserved under dynamics. There might be 3 possibilities: $\rho(A,1)<1/2$, $\rho(A,1)>1/2$ and $\rho(A,1)=1/2$. We start from the first case. Clearly $\rho(A,1)<1/2$ yields $\rho(A,00)>0$. Consider a configuration $x:=\langle A\,\underbrace{11\cdots1}_{2|A|}\rangle$, where $x=\langle B\rangle\equiv$ $\ldots BBB\ldots$ stays for a space-periodic configuration. By the construction $\rho(x,1)$ $=(\rho(A,1)\cdot|A|+2|A|)/(3|A|)\geqslant 2/3>1/2$. Therefore $T^t x\xrightarrow{t\to\infty}\text{Free}^*$ and hence $\rho(T^t x,00)\xrightarrow{t\to\infty}0$. Assume now that the density is preserved, i.e., $\rho(T^t x,A)=\rho(x,A)\ \forall t$. Then by Lemma 2.2 we have

$$\rho(T^t x,00)\geqslant\rho(T^t x,A)\cdot\rho(A,00)=\rho(x,A)\cdot\rho(A,00)>0,$$

while the left hand side vanishes when $t\to\infty$. We came to a contradiction.

If $\rho(A,1)>1/2$ we shall follow a similar argument, considering another space-periodic configuration $x:=\langle A\,\underbrace{00\cdots0}_{2|A|}\rangle$.

In a more delicate case $\rho(A,1)=1/2$ we do the following. If additionally $\rho(A,11)>0$ we follow the same argument as in the case $\rho(A,1)$ $<1/2$ to show that $\rho(x,11)>0$, while $\rho(T^t x,11)\xrightarrow{t\to\infty}0$. If $\rho(A,00)>0$ we follow the case $\rho(A,1)>1/2$ to show that $\rho(x,00)>0$, while $\rho(T^x,00)\xrightarrow{t\to\infty}0$. It remains to consider the case when $\rho(x,11)=\rho(x,00)$ $=0$, i.e., $A=1010\ldots10$ or $A=0101\ldots01$. In the first of these cases we choose $x:=\langle 1A0\rangle$. Then

$$\begin{aligned}\rho(\langle 1A0\rangle,A)&=\lim_{n\to\infty}\frac{n}{n(|A|+2)}=\frac{1}{|A|+2}<\frac{1}{2}\\&=\lim_{n\to\infty}\frac{n|A|/2}{n|A|}=\rho(T(\langle 1A0\rangle),A).\end{aligned}$$

In the second case we choose $x:=\langle 0A1\rangle$ to come to a similar contradiction. ∎

Proof of Theorem 1.3 in the Case v=M=1. The proof follows from Lemmas 2.8–2.12. ∎

Proof of Theorem 1.4 in the Case v=M=1. We have the following identity: $\rho_\pm(x,1)=\rho_\pm(x,10)+\rho_\pm(x,11)$. If $\rho_+(x,1)\leqslant 1/2$ then $T^t x\to\text{Free}$ and $\rho_+(T^t x,11)\to 0$, thus $\Phi_\pm(T^t x)=\rho_\pm(T^t x,10)=\rho_\pm(T^t x,1)-\rho_\pm(T^t x,11)$ $\to\rho_\pm(T^t x,1)=\rho_\pm(x,1)$. The situation $\rho_-(x,1)\geqslant 1/2$ can be reduced to the previous one by going to the dual configuration.

Consider now the case $\rho_-(x,1)<1/2<\rho_+(x,1)$. By definition there exists a sequence of pairs of positive integers $n_i',m_i'\to\infty$ such that $\rho(x[-n_i',m_i'],1)\xrightarrow{i\to\infty}\rho_-(x,1)<1/2$. For each i we choose integers $n_i\geqslant n_i'$, $m_i\geqslant m_i'$ to be the smallest integers satisfying the condition that

$-n_i - 1$ is the ending point and $m_i + 1$ is the starting point of some nonoverlapping minimal intervals of the configuration x. If there are no more nonoverlapping minimal intervals in the considered direction or the segment $x[-n_i', m_i']$ intersects with an infinitely long minimal interval we set $n_i := n_i'$ or $m_i := m_i'$ respectively, depending on the direction where this event occurs. Clearly, we have $\rho(x[-n_i, m_i], 1) \leqslant \rho(x[-n_i', m_i'], 1)$ and thus $\rho(x[-n_i, m_i], 1) \xrightarrow{i \to \infty} \rho_-(x, 1) < 1/2$. By the definition of minimal intervals after $t_i := (n_i + m_i)/2 + 1$ iterations all clusters of particles inside of the segment $x[-n_i, m_i]$ will disappear and all particles will become free. Therefore we can again apply the same argument as in the case $\rho_+(x, 1) \leqslant 1/2$ and obtain the relation for the lower limit of the flux. To obtain the relation for the upper limit one should consider the dual configuration. ▮

Lemma 2.14. Let $x \in X$ satisfy the assumption that there exists a number $\gamma \in (0, 1)$ such that $\forall n \in \mathbb{Z}_+$ and for any word $A \subseteq x[-n, n]$ with $|A| > 2\gamma n$ we have $\rho(A, 1) \leqslant 1/2$. Then $\mathrm{dist}(T^t x, \mathrm{Free}) \leqslant 2^{-t/\gamma+1}$ for any $t \in \mathbb{Z}_+$. If x^* satisfies the same assumption, then we have $\mathrm{dist}(T^t x, \mathrm{Free}^*) \leqslant 2^{-t/\gamma+1}$.

Proof. Consider only those $n \in \mathbb{Z}_+$ for which the largest minimal word containing a cluster of particles in $x[-n, n]$ also belongs to $x[-n, n]$. By the assumption of Lemma the length of the largest minimal interval containing in the segment $x[-n, n]$ does not exceed $2\gamma n$. Therefore the corresponding clusters of particles with disappear after at most γn iterations, and thus for all sufficiently large $t \in \mathbb{Z}_+$ all particles in the segment $T^t x[-t/\gamma, t/\gamma]$ will become free. Thus the closest to the origin nonfree particle can appear not earlier as at the site t/γ, which gives the desired estimate of the rate of convergence. The second statement follows from the same argument applied to the dual map. ▮

Lemma 2.15. Let $x \in X$ satisfy the same assumption as in Lemma 2.14, then

$$\limsup_{n \to \infty} \frac{1}{2n} \sum_{i=-n}^{n} V(T^n x, 1) = F_{1,1}(\rho(x, 1)),$$

where the function $F_{v,M}$ is defined in the formulation of Theorem 1.4.

Proof. Observe that $\frac{1}{2n} \sum_{i=-n}^{n} V(x, 1) = \rho(x[-n, n], 10)$. Applying the same argument as in the proof of Lemma 2.14 we see that after n iterations the segment $T^n x[-n/\gamma, n/\gamma]$ contains only free particles. Therefore $\rho(x[-n, n], 10) = \rho(x[-n, n], 1)$, which yields the desired equality. ▮

Corollary 2.16. The statements of Lemmas 2.14, 2.15 remain valid if instead of $\forall n \in \mathbb{Z}_+$ we assume that n belongs to the subset of $\mathbb{Z}_+$ of density 1.

Lemma 2.17. $\forall \gamma \in (0, 1)$ for μ_p-a.a. configurations $x \in X$ the set of $n \in \mathbb{Z}_+$, for which any word $A \subseteq x[-n, n]$ with $|A| > 2\gamma n$ satisfies the inequality $\rho(A, 1) \leqslant 1/2$, has the density 1.

Proof.[4] Let $\{x_i\}_{-\infty}^{\infty}$ be a Bernoulli sequence with the density $\mathscr{P}(x_i = 1) = p < 1/2$ for all $i \in \mathbb{Z}$. Introduce a sequence of functions $y_n(\tau) := \frac{1}{2n+1} \sum_{i=-n}^{-n+\lfloor 2n\tau \rfloor} x_i$ depending on a real variable $\tau \in [\gamma, 1]$, and consider a functional

$$\phi(y(\tau)) := \sup_{\tau \in [0, 1-\gamma]} \sup_{\gamma \leqslant s \leqslant 1-\tau} \frac{1}{s} (y(\tau+s) - y(\tau))$$

defined in Skorohod space of functions $y(\tau)$. Then the quantity under question is the probability $\mathscr{P}(\phi(y_n(\tau)) \leqslant 1/2 \ \forall \tau \in [\gamma, 1])$. Since $y_n(\tau)$ converges in probability for a given τ to $\tilde{y}(\tau) := p\tau$ and the functional ϕ is continuous, $\phi(y_n(\tau))$ converges to $\phi(\tilde{y}(\tau))$ (functional law of large numbers). Thus we have

$$\mathscr{P}(\phi(y_n(\tau)) \leqslant 1/2 \ \forall \tau \in [\gamma, 1]) \to \mathscr{P}(\phi(\tilde{y}(1)) \leqslant 1/2) = 1,$$

where the rate of convergence $(\mathscr{P}(\phi(y_n(1)) > 1/2))^{1/n} \xrightarrow{n \to \infty} \sqrt{2p(1-p)}$ follows by the combination of the large deviation principle for the functions $y_n(\tau)$ and the contraction principle (see, e.g., ref. 6). ∎

Corollary 2.18. Results of Lemmas 2.14, 2.15, 2.17 prove Theorem 1.5 in the case $v = M = 1$. Since we shall show later that the analysis of $T_{v,M}$ in all cases can be reduced to the case of $v = M = 1$ we shall not consider consider the proof of this result for other cases.

3. THE ONE LANE FAST PARTICLES MODEL (T_v, X)

Note that the analysis of dynamics of the slow particles model (T, X) is divided logically into two parts: first, we study low density initial configurations $x \in X$ with $\rho_+(x, 1) \leqslant 1/2$, and then for high density configurations $x \in X$ with $\rho_-(x, 1) > 1/2$ we pass to the dual ones using the property that $\rho_+(x^*, 1) \leqslant 1 - \rho_-(x, 1) < 1/2$ and argue that the dual map $T^* \equiv T_{-1}$

[4] The idea of this construction, based on the large deviation principle, was proposed by A. Puhal'skii.

has exactly the same asymptotic properties as T. The problem with the fast particles model (T_v, X) is that the dual map $T_v^* \neq T_{-v}$ in this case, and, in fact, has a very nontrivial dynamics. Namely, T_{-v}^* corresponds to the situation, known in physical literature (in the case $v=2$) as a traffic model with "smart drivers," who anticipating the motion of at most v cars ahead, may move to an occupied site ahead of it with the maximal velocity 1. Example for the case $v=2$: $\langle 01110\rangle \xrightarrow{T_2} \langle 01011\rangle$.

Therefore since we are unable to study directly the dual map in this case and according to the entire ideology of this paper, we elaborated a reduction to the main case $v=M=1$ based on the following consideration. Note that under the action of the map T on $x \in X$ each pair 10 goes to 01 (i.e., the position of a particle and a hole are exchanged). Therefore T is equivalent to the substitution rule $10 \to 01$. To apply this idea to the case of T_v we introduce an alphabet $\mathscr{A}_v := \{0_1, 0_2, ..., 0_v, 1\}$ with $v+1$ symbols and a map $C_v\colon X \to \mathscr{A}_v^{\mathbb{Z}} \equiv X_v$ defined as follows: for each segment $x[i, i+n+1] = 1\,\underbrace{0\cdots 0}_{n}\,1$, we set $C_v x[i, i+n+1] := 1\,\underbrace{0_v\cdots 0_v}_{\lfloor n/v\rfloor}\,0_{n-\lfloor n/v\rfloor v}1$. If $n-\lfloor n/v\rfloor v=0$ we shall drop the last element in $C_v x[i, i+n+1]$. It remains to define the action of C_v on "tails" of x consisting of only zeros, which we set according to the following rules: $...0001... \xrightarrow{C_v} ...0_v0_v0_v1...$ and $...1000... \xrightarrow{C_v} ...10_v0_v0_v....$

Now we are ready to define the substitution map $S_v\colon X_v \to X_v$ acting in the set X_v according to the set of v substitution rules $10_i \to 0_i1$ for $0<i\leqslant v$, which generalizes the substitution rule for the slow particles dynamics in the case of v different types of holes.

To study the life-time of clusters of particles in configurations $x \in X_v$ we introduce also a new map $\tilde{T} := C_v C_v^{-1} S_v$ from the space X_v into itself.

Lemma 3.1. $T_v = C_v^{-1} S_v C_v$, and $T_v^n = C_v^{-1} \tilde{T}^n C_v$ for any $n \in \mathbb{Z}_+$.

Proof. Straightforward. Note only that the map $C_v C_v^{-1}$ needs not to be identical, for example:

$$C_v C_v^{-1}\langle 10_{v-1}0_1 1\rangle = \langle 10_v 1\rangle,$$

and that $C_v^{-1} C_v C_v^{-1} \equiv C_v^{-1}$ which we use to get the 2nd relation. ▮

Observe that the map $\tilde{T}$ acts on X_v in exactly the same way as T acts on the space of binary sequences, namely $\tilde{T}$ moves each particle by one position forward if there is no particle there or the particle preserves its position otherwise. So the only difference is that now we have v different types of zeros, instead of the only one type in the case $v=1$.

$$
\begin{array}{rccccccccc}
A = & 0_2 & 0_2 & 0_1 & 1 & 0_1 & 1 & 0_1 & 1 & 1 \\
\Gamma^1 A = & & & 0_2 & 0_2 & 1 & 0_1 & 1 & 1 & \\
\Gamma^2 A = & & & 0_1 & 0_2 & 1 & 1 & & & \\
\Gamma^3 A = & & & & 0_2 & 1 & & & &
\end{array}
$$

Fig. 3. Example of the action of Γ^t on Ω^9 with $v=2$.

Therefore to study the life-time of a cluster of particles we apply a similar machinery as in the case of the slow particles model. Denote

$$\mathrm{Ind}(a) := \begin{cases} -v & \text{if } a=1 \\ i & \text{if } a=0_i \end{cases}, \qquad I(A,i) := \max\left\{k<i : \sum_{j=k+1}^{i} \mathrm{Ind}(A_j)<0\right\},$$

$$\Omega^n := \{A \in \mathscr{A}_M^n : A_n = 1, I(A,n)=1\},$$

where A_j is (as usual) the jth element of the word A. Consider a map Γ defined on words of length $n \in \mathbb{Z}_+$ as follows: $(\Gamma A)_i := (\tilde{T} A)_{i+1}$ for all $i=1,2,\ldots,n-1$ (see Fig. 3[5]).

Lemma 3.2. $\Gamma\colon \Omega^n \to \Omega^{n-2-\xi}$, where $0 \leqslant \xi < n-1$. The life-time of the cluster of particles in the end of a word $A \in \Omega^n$ is equal to $(\rho(A,1)\cdot|A|-1)$.

Proof. The proof follows from the same argument as the one of Lemma 2.4. The only difference is that due to the action of Γ the number of elements in ΓA may become smaller than $|A|-2$, since the action of $C_v C_v^{-1}$ may decrease the number of 0_i. On the other hand, during one iteration of the map Γ only one element 1 disappears from (the right hand side) of A, i.e., $\rho(A,1)\cdot|A| = \rho(\Gamma A,1)\cdot|\Gamma A|+1$. Therefore the number of iterations needed for the cluster of particles in the end of the word A to disappear is equal to the number of ones in the word A minus one. ∎

Proof of Theorems 1.2–1.4 for the Case $v>1$, $M=1$. The proof follows immediately from Lemmas 3.1 and 3.2 and the reduction to the case $v=1$ obtained there. ∎

Consider now a special case of *superfast* particles corresponding to the choice of maximal velocity $v=\infty$. Denote

$$X^{(\infty)} := \{x \in X : \forall n \in \mathbb{Z}\ \exists m, m' > |n| : x_m = 1, x_{-m'} = 0\},$$

[5] Note that $0_1 0_2$ in the beginning of the line for $\Gamma^2 A$ is the result of the application of $C_v C_v^{-1}$ and without it the line would start from $0_2 0_1$.

i.e., the set of binary configurations having no infinitely long right "tails" of zeros or left "tails" of ones. Then the maps $T_\infty, T^*_\infty : X^{(\infty)} \to X^{(\infty)}$ are well defined. The substitution rule $1\underbrace{0\cdots 0}_{i}1 \to 10_i 1\ \forall i \in \mathbb{Z}_+$ maps $X^{(\infty)} \to X_\infty$. Strictly speaking, the latter has an infinite alphabet, however all arguments applied in the case of finite v work as well. Moreover here the situation is even simpler, because between each pair of consecutive ones there is only one zero with a certain finite index: $...10_i 10_j...$. Thus the dynamics of $(T_\infty, X^{(\infty)})$ is equivalent to the dynamics of free particles, which gives the flux $\Phi_\pm(x) = 1-\rho_\pm(x, 1)$.

4. REDUCTION OF $T_{1,M}$ TO THE DIRECT PRODUCT OF M MAPS T AND THE GENERAL MULTI LANE MODEL $(T_{v,M}, X_M)$

The model of a multi lane flow of slow particles on a finite lattice has been introduced in ref. 15 and generalized for the case of an infinite lattice $\mathbb{Z}$ in ref. 4, where statistical properties of regular initial configurations have been obtained. However the approach used in ref. 4 does not allow to study the dynamics of general initial configurations, which we shall consider in this Section using a completely different method.

Our first aim is to redistribute a configuration $x \in X_M$ into M binary configurations $\{x^{(j)} \in X = X_1\}_{j=1}^M$, such that $T^t_{1,M}x = \sum_j T^t x^{(j)}$ for all $t \in \mathbb{Z}_+ \cup \{0\}$, where the notation $x = \sum_j x^{(j)}$ means that $x_i = \sum_j x_i^{(j)}$ for each $i \in \mathbb{Z}$. To solve this problem we introduce a *sawtooth redirection* $S_\ell: X_M \to (X_1)^M$ with $S_\ell x = \{x^{(j)}\}_{j=1}^M$ of a configuration $x \in X_M$ to a collection of binary configurations $\{x^{(j)}\}_{j=1}^M$ with the starting point at site $\ell \in \mathbb{Z}$:

$$x_i^{(j)} := \begin{cases} 1 & \text{if } \ i \geqslant \ell \ \text{ and } \ j \in \left(\bigoplus_{k=\ell}^{i-1} x_k, \bigoplus_{k=\ell}^{i} x_k \right] \\ 1 & \text{if } \ i < \ell \ \text{ and } \ j \in \left(\bigoplus_{k=i}^{\ell-1} (-x_k), \bigoplus_{k=i+1}^{\ell-1} (-x_k) \right] \\ 0 & \text{otherwise,} \end{cases}$$

where $a \oplus b := (a+b-1) (\bmod M)+1$ and $\bigoplus_{i=n}^m x_i := x_n \oplus \cdots \oplus x_m$. In other words, for the configuration $x \in X_M$ we construct a bi-infinite "staircase" starting from the site l with the ith stair of height x_i and then redistribute the result modulo M (preserving the site number) among M binary configurations $\{x^{(j)}\}_{j=1}^M$.

With some abuse of notation we shall refer to $(S_\ell x)^{(j)} \equiv x^{(j)}$ as the jth lane of $S_\ell x = \{x^{(j)}\}_{j=1}^M$ and denote the action of the direct product of maps T_v applied at $S_\ell x$ as $T_v S_\ell x := \{T_v x^{(j)}\}_{j=1}^M$.

Example for the case $v=1$, $M=3$ and the starting site ℓ corresponding to the 3d occurrence of "1":

$$S_\ell(\ldots 11211221\ldots) = \begin{matrix}\ldots 10100110\ldots \\ \ldots 00101010\ldots \\ \ldots 01010101\ldots\end{matrix} \xrightarrow{T_{1,3}} \begin{matrix}\ldots 0\ 1010101\ldots \\ \ldots *\ 0010101\ldots \\ \ldots *\ 010101 *\ldots\end{matrix} = S_{\ell+1}(\ldots *\ 112121 *\ldots),$$

where the unknown positions are marked by $*$. Consider also two more examples of the sawtooth redirection for space-periodic configurations in X_3 (in both cases the starting site ℓ corresponds to the first site of the period):

$$S_\ell(\langle 11322\rangle) = \begin{matrix}\langle 00111\rangle \\ \langle 01101\rangle \\ \langle 10110\rangle\end{matrix} \qquad S_\ell(\langle 2312\rangle) = \begin{matrix}\langle 011011011101\rangle \\ \langle 110101101101\rangle \\ \langle 110111010110\rangle\end{matrix}$$

Observe that after the redirection the length of the spatial period might change drastically (2nd example).

Symbolically the sawtooth redirection is shown in Fig. 4(b) by curvilinear lines corresponding to sawtooth rows of ones, open circles mark the intersections of these lines with the "lanes" j, j', i.e., the positions where $x^{(j)}$ or $x^{(j')}$ are equal to 1 (all other positions on these lanes are occupied by zeros). Note that in the case of two lanes (i.e., $M=2$) the redirection between lanes has been considered in ref. 2, however the general case $M>2$ turns out to be much more delicate and cannot be obtained as a straightforward generalization of the procedure in ref. 2.

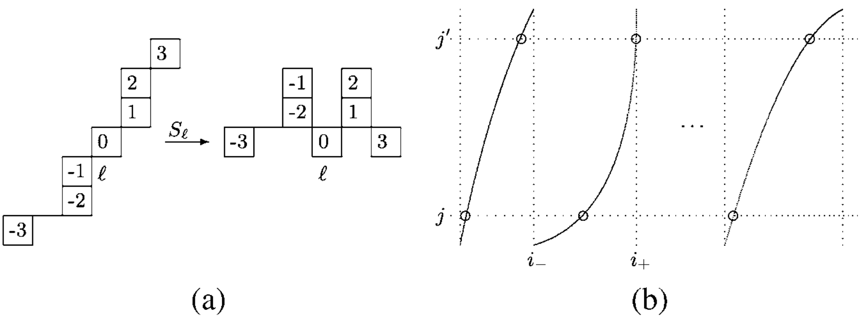

Fig. 4. "Sawtooth redirection:" (a) action of the map S_ℓ on individual particles marked by squares with their relative numbers ($M=3$); (b) symbolical representation of $S_\ell x$, $[i_-, i_+]$ – one of intervals of monotonicity.

Theorem 4.1. For any $x \in X_M$ and $\ell \in \mathbb{Z}$ and $S_\ell(x) \equiv \{x^{(j)}\}_{j=1}^{M}$ we have

(a) $x = \sum_j (S_\ell x)^{(j)}$,

(b) $|\rho(x^{(j)}[n+1, n+k], 1) - \rho(x^{(j')}[n+1, n+k], 1)| \leqslant 1/k$ $\forall j, j' \in \{1,..., M\}$, $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$,

(c) $S_{\ell+k}x = \{(S_\ell x)^{(j \oplus k)}\}_{j=1}^{M}$ $\forall k \in \mathbb{Z}_+$,

(d) $\forall v \geqslant 1$ we have $T_v S_\ell x = S_{\ell+\xi} x'$ for some $\xi \in \{0, 1,..., v\}$ and $x' \in X_M$ which doesn't depend on ℓ,

(e) $T S_\ell x = S_{\ell+\xi}(T_{1,M} x)$ for some $\xi \in \{0, 1\}$.

Proof. The statement (a) follows immediately from the definition of the sawtooth redirection, because during the redirection each particle preserves its position i.

The property (b) is equivalent to the assumption that

$$\left| \sum_{i=1}^{k} x_{n+i}^{(j)} - \sum_{i=1}^{k} x_{n+i}^{(j')} \right| \leqslant 1,$$

i.e., that the number of particles in the same segment of different "lanes" j, j' can differ at most by 1. According to the "sawtooth redirection" for any given finite segment of integers $n+1, n+2,..., n+k$ the number of intersections of the curvilinear lines in Fig. 4(b) with the horizontal line at height j differs from number of intersections with the horizontal line at height j' at most by one. This immediately yields the property (b).

The collection of binary configurations $S_\ell(x)$ has a row of ones at site i of height k if and only if $x_i = k$, and the change of the starting point ℓ of the redirection only changes cyclically the starting point 1 of the enumeration of lanes $x^{(j)}$. This proves the property (c).

Observe now that the definition of $S_\ell(x)$ is equivalent to the existence of a partition of $\mathbb{Z}$ into segments $[i_-, i_+]$ such that $x_{i_-}^{(1)} = 1$, $x_{i_+}^{(M)} = 1$ (except for the most left segment where $i_- = -\infty$ and the most right one where $i_+ = \infty$) and for any $1 < j < M$ there exists the only one $i \in [i_-, i_+]$ such that $x_i^{(j)} = 1$. Indeed, according to the "sawtooth redirection" the curvilinear lines in Fig. 4(b) have the property that the intersection with the horizontal line at height j occurs not earlier than with the horizontal line at height $j' > j$ (the curvilinear lines may have vertical segments). To simplify the notation we shall say that $S_\ell(x)$ is monotonous on $[i_-, i_+]$.

Consider the interval of monotonicity $[i_-, i_+]$ which starts from ℓ, i.e., $i_- = \ell$. We set ξ to be equal to the minimum of the number of not occupied positions in $x^{(1)}$ ahead of the site i_- (which is occupied by 1). Then under the action of T_v the particle at the site i_- of the 1-st lane moves by ξ positions to the right. Observe that all particles on the other lanes in the segment $[i_-, i_+]$ have at least ξ not occupied positions ahead of them, and therefore all these particles will move at least ξ positions to the right. Thus to prove that the monotonicity is preserved it is enough to note that the particle on the lane M cannot move further to the right than the first particle on the first lane of the next interval of monotonicity. Indeed, the latter is a trivial consequence of the definition of intervals of monotonicity. This finishes the proof of the statement (d) except the last part, which follows from the statement (c).

To prove the statement (e), observe that by the definition of the map $T_{1,M}$ (see Section 1) a particle at the site i of the lane j can switch to the lane j' if and only if $x^{(j)}[i, i+1] = 11$ and $x^{(j')}[i, i+1] = 00$, which contradicts to the definition of the intervals of monotonicity. Therefore under the sawtooth redirection no particle in $S_\ell(x)$ will change its lane. ∎

Corollary 4.2. The sawtooth redirection gives a simple constructive way to rearrange vehicles in a multi lane traffic flow between lanes (preserving their positions in the flow) in order to achieve the maximal available flux.

According to Theorem 4.1(d) the map $x \to \sum_j T_v(S_\ell x)^{(j)}$ is well defined as a map from X_M into itself and does not depend on the choice of the starting site $\ell \in \mathbb{Z}$. Moreover, it can be shown that this formula coincides with (1.1) in the case $v = 1$, and it clearly coincides with T_v in the case $M = 1$. Therefore we use this relation as a definition of the dynamics of a general multi lane flow in the case $v, M > 1$, namely we set $T_{v,M} x := \sum_j T_v(S_0 x)^{(j)}$.

Proof of Theorem 1.2 for the Case v, $M \geqslant 1$ and $A \subset X$. The proof follows now from the sawtooth redirection, which gives the reduction to the one-lane case. It remains to show that the statistics of more general words $A \subset X_M$ with $|A| = 1$ might be not preserved under dynamics. The reason for this is that if $M > 1$ the multiplicities might be not preserved. Indeed, let $a \in \mathscr{A}_M \setminus \{0, 1\}$. Then $\rho(\langle a(M-a+1)\, 0\rangle, a) = \frac{1}{3}(1 + \lfloor (M-a+1)/a \rfloor + 0)$, while

$$
\begin{aligned}
\rho(T_{1,M}\langle a(M-a+1)\, 0\rangle, a) &= \rho(\langle 1(a-1)(M-a+1)\rangle, a) \\
&= \tfrac{1}{3}(0 + 0 + \lfloor (M-a+1)/a \rfloor + 0) \\
&< \rho(\langle a(M-a+1)\, 0\rangle, a). \quad ∎
\end{aligned}
$$

Proof of Theorems 1.3 and 1.4 for the Case v, M > 1. Consider a configuration $x \in X_M$. According to Theorem 4.1(b) for $S_0 x \equiv \{x^{(j)}\}_{i=1}^{M}$ we have $\forall n, m \in \mathbb{Z}_+$ that

$$|\rho(x^{(j)}[-n, m], 1) - \rho(x^{(j')}[-n, m], 1)| \leqslant \frac{1}{m+n+1}.$$

Thus going to the limit as $n, m \to \infty$ and using Theorem 4.1(a) we get $\rho_{\pm}(x^{(j)}, 1) = \frac{1}{M} \rho_{\pm}(x, 1)$ for each $j \in \{1, \ldots, M\}$. Therefore the application of the results obtained in Sections 2 and 3 in the case of one-lane flows (i.e., in the case of the map T_v) proves the statements under question. ▮

5. DYNAMICS OF MEASURES AND CHAOTICITY

In this section we shall study the action of the map $T_{v,M}$ in the space $\mathcal{M}(X_M)$ of probabilistic measures on X_M. This action is defined as follows: $T_{v,M}\mu(Y) := \mu(T_{v,M}^{-1} Y)$ for a measure $\mu \in \mathcal{M}(X_M)$ and a measurable subset $Y \subseteq X_M$. A measure $\mu \in \mathcal{M}(X_M)$ is called translation invariant if it is invariant with respect to the action of the shift map $\sigma \colon X_M \to X_M$.

Lemma 5.1. If $\mu \in \mathcal{M}(X_M)$ is translation invariant then this property holds for $T_{v,M}^t \mu\ \forall t \in \mathbb{Z}_+$.

Proof. We have $T_{v,M}^t \mu(Y) = \mu(T_{v,M}^{-t} Y) = \mu(\sigma T_{v,M}^{-t} Y) = \mu(T_{v,M}^{-t} \sigma Y) = T_{v,M}^t \mu(\sigma Y)$. ▮

One might expect that under the action of the map $T_{v,M}$ any translation invariant measure should converge to a Bernoulli one. Indeed,

$$\begin{aligned} T\mu_p(x \in X : x_0 = 1) &= \mu_p(x \in X : x[0, 1] = 11) + \mu_p(x \in X : x[-1, 0] = 10) \\ &= \mu_p(x \in X : x[0, 1] = 11) + \mu_p(x \in X : x[0, 1] = 10) \\ &= \mu_p(x \in X : x_0 = 1). \end{aligned}$$

On the other hand, the product structure is not preserved even in the case of the model of slow particles.

Lemma 5.2. The measure $T\mu_p$ is not a product one for any $0 < p < 1$.

Proof. We have

$$\begin{aligned} T\mu_p(x \in X : x[0,1] = 11) &= \mu_p(x \in X : x[0,2] = 111) \\ &\quad + \mu_p(x \in X : x[-1,2] = 1011) \\ &= p^3 + p^3(1-p) = p^3(2-p) \\ &\neq p^2 = \mu_p(x \in X : x[0,1] = 11). \end{aligned}$$

Thus the measure $T\mu_p$ does not have the product structure. ▮

It is of interest that in the case $v > 1$ even the average value $\mu_p(x \in X : x_0 = 1)$ is not preserved under dynamics. Indeed,

$$\begin{aligned} T_v\mu_p(x \in X : x_0 = 1) &= \mu_p(x \in X : x[0,1] = 11) \\ &\quad + \sum_{i=1}^{v} \mu_p(x \in X : x[-1,0] = 10_i) \\ &= p^2 + p(1-p) + \cdots + p(1-p)^v = p + p \sum_{i=2}^{v} (1-p)^i \\ &> p = \mu_p(x \in X : x_0 = 1). \end{aligned}$$

Note that in the case of the slow particles model $(T_{1,1}, X)$ some results about the set of $T_{1,1}$-invariant measures and mathematical expectations of the limit flux with respect to them were studied in ref. 2.

In ref. 3 it has been proven that the dynamical system $(T_{1,M}, X_M)$ is chaotic in the sense that its topological entropy is positive. Moreover this paper gives an asymptotically exact (as $M \to \infty$) representation for the entropy. The extension of this result to the case $(T_{v,M}, X_M)$ with $v > 1$ is straightforward.

6. A MODEL OF A PEDESTRIAN GOING IN A SLOWLY MOVING CROWD

Obtained results make it possible to explain the following practical observation: it turns out that a pedestrian going in a slowly moving crowd may go faster against the "flow" than in the same direction as other people go. This observation certainly contradicts to standard probabilistic models describing a diffusion of particles along/against the flow and indicates a special (nonrandom) intrinsic structure of the flow in the case under consideration. In this section we shall consider a model of this process described as a passive tracer in the 1-lane flow of fast particles.

Let $T_v^t x$, $v \geqslant 1$ describe the 1-lane flow of particles and let the passive tracer occupy the position i at time t. Then before carrying out the next time step of the model describing the flow of particles, the tracer moves in its chosen direction to the closest (in this direction) position of a particle of the configuration $T_v^t x$. For example, if the moving forward tracer occupies the position 2 and the closest particle in this direction occupies the position 5, then the tracer moves to the position 5. After that the next iteration of the flow occurs, the tracer moves to its new position, etc.

To be precise, let us fix a configuration $x \in X$ with $\rho_-(x, 1) > 0$ and introduce the maps $\tau_x^{\pm}: \mathbb{Z} \to \mathbb{Z}$ defined as follows:

$$\tau_x^+ i := \min\{j: i < j, x_j = 1\}, \qquad \tau_x^- i := \max\{j: i > j, x_j = 1\}.$$

Then the simultaneous dynamics of the configuration of particles (describing the flow) and the tracer is defined by the skew product of two maps—the map T_v and one of the maps $\tau_{\cdot}^{\pm}$, i.e.,

$$(x, i) \to \mathscr{T}_{\pm}(x, i) := (T_v x, \tau_x^{\pm} i),$$

acting on the extended phase space $X \times \mathbb{Z}$. The sign $+$ or $-$ here corresponds to the motion along or against the flow. We define the *average (in time) velocity* of the tracer $V(t, x)$ as $S(t)/t$, where $S(t)$ denotes the total distance covered by the tracer (which starts at the site 0) up to the moment t with the positive sign if the tracer moves forward, and the negative sign otherwise.

Theorem 6.1. Let $x \in \{x \in X : \operatorname{dist}(T_v^t x, \text{Free} \cup \text{Free}^*) \leqslant 2^{-t/\gamma+1}\}$ for all $t \in \mathbb{Z}_+$ and some $0 < \gamma < 1$. If $0 < \rho_+(x, 1) \leqslant \frac{1}{v+1}$, then $V(t, x) \xrightarrow{t \to \infty} v$ if the tracer moves along the flow (i.e., in the case $\mathscr{T}_+$), and $\lim_{t \to \infty} \left(\begin{smallmatrix}\sup\\ \inf\end{smallmatrix}\right) V(t, x) = \frac{-1}{\rho_{\pm}(x, 1)} + 1$ in the opposite case. If $\rho_-(x, 1) > 1 - \frac{1}{v+1}$ and the tracer moves against the flow then $V(t, x) \xrightarrow{t \to \infty} -1$.

Remark. The assumption about the initial configurations is satisfied for μ_p-a.a. $x \in X_M$ (see Theorem 1.5).

Proof. Since we assume that $T_v^t x$ converges to the attractor $\text{Free} \cup \text{Free}^*$ with the exponentially fast rate, then at the moment $t \in \mathbb{Z}_+$ we have an exponentially long (in t) interval of the configuration $T_v^t x$ consisting of only free particles or free holes (depending on the density). As we shall show that $V(t, x)$ converges to a constant, then to study its value we can restrict the analysis to the case $x \in \text{Free} \cup \text{Free}^*$.

Under the assumption $0 < \rho_+(x,1) \leqslant \frac{1}{v+1}$ we have $T_v^t x \xrightarrow{t\to\infty} \mathrm{Free}_v$. In the case of $\mathcal{T}_+$ the tracer will run down one of the particles and will follow it, but cannot outstrip. Indeed after each iteration of the flow this free particle occurs exactly v positions ahead of the tracer. Thus $V(t,x) \xrightarrow{t\to\infty} v$.

Consider now the case when the tracer moves backward with respect to the flow. Then each time when the tracer encounters a particle, on the next time step this particle moves in the opposite direction and does not interfere with the movement of the tracer. We assume again that $x \in \mathrm{Free}_v$ and consider the case $0 < \rho_+(x,1) \leqslant \frac{1}{v+1}$. If on the spread of length n there are m particles, i.e., m obstacles for the tracer then the average velocity on this segment is equal to $\frac{n-m}{m}$. Going to the limit as $n \to \infty$ we obtain the desired estimate.

It remains to consider the case $\rho_-(x,1) > 1 - \frac{1}{v+1}$ and thus $T_v^t x \xrightarrow{t\to\infty} \mathrm{Free}_v^*$, i.e., to the flow where all holes move at maximal velocity $-v$. Thus after each iteration the tracer moves exactly by one position to the left (since it never can encounter a hole), which gives the limit velocity -1. ∎

Observe that the motion against the flow is efficient only in the case of low density of particles when $\rho_+(x,1) \leqslant \frac{1}{v+1}$. On the other hand, in the high density region in the case of the motion along the flow and in the region $\frac{1}{v+1} < \rho_-(x,1) < 1 - \frac{1}{v+1}$ in the case of the motion against the flow the limit velocity of the tracer depends not only on the densities, but also on the fine structure of the configuration x. Moreover, this concerns also the case of "untypical" initial configurations with $0 < \rho_-(x,1) < 1/2 < \rho_+(x,1)$, when there might be arbitrary long (even infinite) minimal words for both particles and holes.

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